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ON (\bar{N}, p_n) SUMMABILITY FACTORS OF INFINITE SERIES

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Abstract

In this paper sufficient conditions have been obtained for $\sum a_n \epsilon_n$ to be summable $|\bar{N}, q_n|$ whenever $\sum a_n$ is bounded (\bar{N}, p_n) .

Key-Words : Infinite Series, .

1. Let Σa_n be a given infinite series with s_n for its nth partial sum. Let $\{t_n\}$ and $\{t'_n\}$ be the sequences of (\bar{N}, p_n) and (\bar{N}, q_n) means of the sequence $\{s_n\}$ respectively. Then

$$(1.1) \quad t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v, (P_n \neq 0),$$

and

$$(1.2) \quad t'_n = \frac{1}{Q_n} \sum_{v=0}^n q_v s_v, (Q_n \neq 0),$$

where

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty \text{ as } n \rightarrow \infty, (P_{-i} = p_{-i} = 0, i \geq 1)$$

and

$$Q_n = \sum_{v=0}^n q_v \rightarrow \infty \text{ as } n \rightarrow \infty, (Q_{-i} = q_{-i} = 0, i \geq 1).$$

A series Σa_n is said to be bounded (\bar{N}, p_n) or $O(1)(\bar{N}, p_n)$ if

$$\sum_{v=1}^n p_v s_v = O(P_n), \text{ as } n \rightarrow \infty.$$

2. Concerning the absolute summability of a series $\sum a_n$ Sunouchi [1] proved that if

$$(2.1) P_n q_n = O(p_n Q_n),$$

then $|\overline{N}, p_n| \Rightarrow |\overline{N}, q_n|$.

In the present paper I have proved that the condition (2.1) is also a sufficient condition for $\sum a_n \varepsilon_n$ to be summable $|\overline{N}, q_n|$ whenever $\sum a_n$ is bounded (\overline{N}, p_n) for suitable restricted $\{\varepsilon_n\}$.

Theorem 1. let $\{p_n\}$ and $\{q_n\}$ be two positive monotonic non-increasing sequences such that $P_n q_n = O(p_n Q_n)$. If $t_n = O(\mu_n)$, as $n \rightarrow \infty$, where $\{\mu_n\}$ is a positive, monotonic non-decreasing sequence, then $\sum a_n \varepsilon_n$ is summable $|\overline{N}, q_n|$, if $\{\varepsilon_n\}$ satisfies the conditions.

$$(2.2) \sum_{n=1}^{\infty} |\varepsilon_n| \mu_n < \infty,$$

$$(2.3) \sum_{n=1}^{\infty} \frac{P_n}{P_{n+1}} \mu_n |\Delta^2 \varepsilon_n| < \infty$$

Remarks : I If I take $q_n = p_n$, for $n = 0, 1, 2, \dots$ in my theorem 1, then the result of Daniel [1] can be obtained as a particular case.

Theorem A [1]. Let $\{p_n\}$ be a positive and monotonic non-increasing sequence. If $t_n = O(\mu_n)$ as $n \rightarrow \infty$, where $\{\mu_n\}$ is a positive, monotonic non-decreasing sequence, then $\sum a_n \varepsilon_n$ is summable $|\overline{N}, p_n|$, if $\{\varepsilon_n\}$ satisfies the conditions

$$(i) \sum_{n=1}^{\infty} |\varepsilon_n| \mu_n < \infty,$$

$$(ii) \sum_{n=1}^{\infty} \frac{P_n}{P_{n+1}} \mu_n |\Delta^2 \varepsilon_n| < \infty$$

II If I take $p_n = \frac{1}{n}$ and $q_n = 1$ I have summability factor theorem for $\sum a_n \varepsilon_n$ to be summable $|\mathcal{C}, I|$ whenever $\sum a_n$ is bonded ($R, \log n, 1$).

3. We need the following lemma for the proof of our theorem 1. Lemma [1]. Let $\{p_n\}$ be a positive and monotonic non-increasing sequence. If $\{\varepsilon_n\}$ is such that

(i) $\Delta \varepsilon_n = o(1)$, as $n \rightarrow \infty$; and

(ii) $\sum P_n / p_{n+1} \mu_n |\Delta^2 \varepsilon_n| < \infty$

Where $\{\mu_n\}$ is a positive and monotonic non-decreasing sequence the

$$\sum_{n=1}^{\infty} \frac{P_n \Delta p_n}{p_n p_{n+1}} \mu_n |\Delta \varepsilon_n| < \infty$$

Proof of the theorem 1 : consider,

4.

$$t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v$$

then

$$(4.1) \quad s_n = \frac{1}{P_n} (P_n t_n - P_{n-1} t_{n-1}) = O\left(\frac{P_n}{P_n} \mu_n\right)$$

Now,

$$T_n = \frac{1}{Q_n} \sum_{v=0}^n q_v \sum_{\mu=0}^v a_\mu \varepsilon_\mu \\ = \frac{1}{Q_n} \left[\sum_{v=0}^{n-1} Q_v (-a_{v+1} \varepsilon_{v+1}) + Q_n \sum_{v=0}^n a_v \varepsilon_v \right], \text{ by using Abel's transformation}$$

$$\Rightarrow T_n - T_{n-1} = -\frac{1}{Q_n} \sum_{v=0}^{n-1} Q_v a_{v+1} \varepsilon_{v+1} + \frac{1}{Q_{n-1}} \sum_{v=0}^{n-2} Q_v a_{v+1} \varepsilon_{v+1} - \sum_{v=0}^{n-1} a_v \varepsilon_v + \sum_{v=0}^n a_v \varepsilon_v$$

$$\Rightarrow T_n - T_{n-1} = -\frac{1}{Q_n} \sum_{v=0}^{n-1} Q_v a_{v+1} \varepsilon_{v+1} + \frac{1}{Q_{n-1}} \sum_{v=0}^{n-1} Q_v a_{v+1} \varepsilon_{v+1}$$

$$= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} Q_v a_{v+1} \varepsilon_{v+1}$$

$$= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} Q_{v-1} a_v \varepsilon_v$$

$$= \frac{q_n}{Q_n Q_{n-1}} \left[\sum_{v=1}^{n-1} s_v \Delta (Q_{v-1} \varepsilon_v) + s_n Q_{n-1} \varepsilon_n \right], \text{ by using Abel's transformation}$$

Therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} |T_n - T_{n-1}| &= \sum_{n=1}^{\infty} \left| \frac{q_n}{Q_n Q_{n-1}} \left[\sum_{v=1}^{n-1} s_v \Delta(Q_{v-1} \epsilon_v) + s_n Q_{n-1} \epsilon_n \right] \right| \\
&= \sum_{n=1}^{\infty} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} s_v \Delta(Q_{v-1} \epsilon_v) + \frac{q_n}{Q_n} s_n \epsilon_n \right| \\
&\leq \sum_{n=1}^{\infty} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} s_v \Delta(Q_{v-1} \epsilon_v) \right| + \sum_{n=1}^{\infty} \left| \frac{q_n}{Q_n} s_n \epsilon_n \right| \\
&= \sum_1 + \sum_2, \text{say}
\end{aligned}$$

first I estimate Σ_2

$$\begin{aligned}
\Sigma_2 &= \sum_{n=1}^{\infty} \left| \frac{q_n}{Q_n} s_n \epsilon_n \right| \\
&= O \left[\sum_{n=1}^{\infty} \frac{q_n P_n \mu_n}{Q_n p_n} |\epsilon_n| \right] \text{as } s_n = O \left(\frac{P_n}{p_n} \mu_n \right)
\end{aligned}$$

$$\Sigma_2 = O \left[\sum_{n=1}^{\infty} |\epsilon_n| \mu_n \right], \text{using } q_n P_n = O(p_n Q_n),$$

$$= O(1)$$

Again

$$\begin{aligned}
\Sigma_1 &= \sum_{n=1}^{\infty} \left| \frac{q_n}{Q_n Q_{n+1}} \sum_{v=1}^{n-1} s_v \Delta(Q_{v-1} \epsilon_v) \right| \\
&= \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \left| \sum_{v=1}^{n-1} s_v \Delta(Q_{v-1} \epsilon_v) \right| \\
&= \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} J_1,
\end{aligned}$$

$$\begin{aligned}
\text{where } J_1 &= \left| \sum_{v=1}^{n-1} s_v \Delta(Q_{v-1} \epsilon_v) \right| \\
&= \left| \sum_{v=1}^{n-1} \frac{P_v t_v - P_{v-1} t_{v-1}}{p_v} \Delta(Q_{v-1} \epsilon_v) \right|
\end{aligned}$$

By Abel's transformation,

$$\begin{aligned}
J_1 &= \left| \sum_{v=1}^{n-2} P_v t_v \Delta \left[\frac{\Delta(Q_{v-1} \epsilon_v)}{p_v} \right] + P_{n-1} t_{n-1} \left[\frac{\Delta(Q_{n-1} \epsilon_n)}{p_n} \right] \right| \\
&\leq \left| \sum_{v=1}^{n-2} P_v t_v \Delta \left[\frac{\Delta(Q_{v-1} \epsilon_v)}{p_v} \right] \right| + \left| P_{n-1} t_{n-1} \left[\frac{\Delta(Q_{n-1} \epsilon_{n-1})}{p_{n-1}} \right] \right| \\
&= J_{11} + J_{12}, \text{ say}.
\end{aligned}$$

But,

$$\begin{aligned}
J_{12} &= \frac{P_{n-1} |t_{n-1}|}{p_{n-1}} |\Delta(Q_{n-1} \epsilon_{n-1})| \\
&= \frac{P_{n-1} |t_{n-1}|}{p_{n-1}} |\epsilon_{n-1} \Delta Q_{n-1} + Q_{n-1} \Delta \epsilon_{n-1}| \\
&\leq \frac{P_{n-1} |t_{n-1}|}{p_{n-1}} [Q_{n-1} |\Delta \epsilon_{n-1}| + q_{n-1} |\epsilon_{n-1}|]
\end{aligned}$$

Therefore :

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} J_{12} &\leq \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \frac{P_{n-1} |t_{n-1}|}{p_{n-1}} [Q_{n-1} |\Delta \epsilon_{n-1}| + q_{n-1} |\epsilon_{n-1}|] \\
&= \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \frac{P_{n-1}}{p_{n-1}} Q_{n-1} |\Delta \epsilon_{n-1}| |t_{n-1}| + \sum_{n=1}^{\infty} \frac{q_n q_{n-1}}{Q_n Q_{n-1}} \frac{P_{n-1}}{p_{n-1}} |t_{n-1}| |\epsilon_{n-1}| \\
&= J_{121} + J_{122}, \text{ say}
\end{aligned}$$

Now,

$$\begin{aligned}
J_{121} &= \sum_{n=1}^{\infty} \frac{q_n}{Q_n} \frac{P_{n-1}}{p_{n-1}} |\Delta \epsilon_{n-1}| |t_{n-1}| \\
&= O \left[\sum_{n=1}^{\infty} \frac{q_n}{Q_n} \frac{P_n}{p_n} |\Delta \epsilon_{n-1}| \mu_{n-1} \right], \quad t_n = O(\mu_n) \text{ and } p_{n-1} \geq p_n \\
&= O \left[\sum_{n=1}^{\infty} |\Delta \epsilon_{n-1}| \mu_{n-1} \right], \text{ since } P_n q_n = O(p_n Q_n) \\
&= O(1), \text{ by hypothesis of the theorem}
\end{aligned}$$

Again

$$\begin{aligned}
J_{12} &= \sum_{n=1}^{\infty} \frac{q_n}{Q_n} \frac{q_{n-1} P_{n-1}}{Q_{n-1} p_{n-1}} |t_{n-1}| |\varepsilon_{n-1}| \\
&= O\left[\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |\varepsilon_{n-1}| \mu_{n-1}\right], \text{ since } q_n P_n = O(p_n Q_n) \text{ and } |t_n| = O(\mu_n). \\
&= O(1), \text{ by condition (2.2) of the theorem.}
\end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} J_{12} = O(1)$$

Now consider

$$\begin{aligned}
J_{11} &= \left| \sum_{v=1}^{n-2} P_v t_v \Delta \left[\frac{\Delta(Q_{v-1} \varepsilon_v)}{p_v} \right] \right| \\
&= \left| \sum_{v=1}^{n-2} P_v t_v \left[\frac{\Delta^2(Q_{v-1} \varepsilon_v)}{p_v} + \Delta \frac{1}{p_v} \Delta(Q_v \varepsilon_{v+1}) \right] \right| \\
&\leq \sum_{v=1}^{n-2} P_v |t_v| \frac{|\Delta^2(Q_{v-1} \varepsilon_v)|}{p_v} + \sum_{v=1}^{n-2} P_v |t_v| |\Delta(Q_v \varepsilon_{v+1})| \frac{\Delta p_v}{p_v p_{v+1}} \\
&= J_{111} + J_{112} \text{ say.}
\end{aligned}$$

Consider,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} J_{112} &= \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v| |\Delta(Q_v \varepsilon_{v+1})| \Delta p_v}{p_v p_{v+1}} \\
&= \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v| |Q_v \Delta \varepsilon_{v+1} + \varepsilon_{v+2} \Delta Q_v| \Delta p_v}{p_v p_{v+1}} \\
&\leq \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v| |Q_v| |\Delta \varepsilon_{v+1}| \Delta p_v}{p_v p_{v+1}} + \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v| |\varepsilon_{v+2}| q_{v+1} \Delta p_v}{p_v p_{v+1}} \\
&= M_1 + M_2, \text{ say.}
\end{aligned}$$

Now

$$\begin{aligned}
M_2 &= \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v| |\epsilon_{v+2}| q_{v+1} \Delta p_v}{p_v p_{v+1}} \\
&\leq \sum_{n=1}^{\infty} \frac{P_{v+1} |t_v| |\epsilon_{v+2}| \Delta p_v q_{v+1}}{p_v p_{v+1}} \sum_{n=v+2}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \\
&= O\left(\sum_{v=1}^{\infty} \frac{P_{v+1}}{p_{v+1}} \frac{q_{v+1}}{Q_{v+1}} \frac{\Delta p_v}{p_v} |\epsilon_{v+2}| \mu_v\right), \text{ since } t_n = O(\mu_n) \\
&= O\left(\sum_{v=1}^{\infty} \frac{\Delta p_v}{p_v} |\epsilon_{v+2}| \mu_v\right), \text{ since } P_v q_v = O(p_v Q_v) \\
&= O\left(\sum_{v=1}^{\infty} |\epsilon_{v+2}| \mu_v\right) = O(1), \text{ by condition (2.2) of the theorem.}
\end{aligned}$$

Hence $M_2 = O(1)$.

Now

$$\begin{aligned}
M_1 &= \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v| |Q_v| |\Delta \epsilon_{v+1}| \Delta p_v}{p_v p_{v+1}} \\
&= \sum_{v=1}^{\infty} \frac{P_v |t_v| |Q_v| |\Delta \epsilon_{v+1}| \Delta p_v}{p_v p_{v+1}} \sum_{n=v+2}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \\
&= O\left(\sum_{v=1}^{\infty} \frac{P_v Q_v |\Delta \epsilon_{v+1}| \Delta p_v \mu_v}{p_v p_{v+1} Q_{v+1}}\right) \\
&= O\left(\sum_{v=1}^{\infty} \frac{P_v \Delta p_v}{p_v p_{v+1}} |\Delta \epsilon_{v+1}| \mu_v\right) = O(1), \text{ by the lemma.}
\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} J_{112} = O(1).$$

Now consider

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} J_{111} = \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-2} \frac{P_v |t_v|}{p_v} \left| \Delta [\Delta(Q_{v-1} \epsilon_v)] \right| \right)$$

$$\begin{aligned}
&\leq \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v|}{p_v} \\
&\quad \left[|\Delta Q_{v-1}| |\Delta \epsilon_v| + Q_v |\Delta^2 \epsilon_v| + q_v |\Delta \epsilon_{v+1}| + \Delta q_v |\epsilon_{v+2}| \right] \\
&= \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v|}{p_v} |\Delta Q_{v-1}| |\Delta \epsilon_v| + \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v|}{p_v} Q_v |\Delta^2 \epsilon_v| \\
&\quad + \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v|}{p_v} q_v |\Delta \epsilon_{v+1}| + \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v|}{p_v} \Delta q_v |\epsilon_{v+2}| \\
&= Y_1 + Y_2 + Y_3 + Y_4, \text{ say}
\end{aligned}$$

$$\begin{aligned}
Y_1 &= \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v|}{p_v} |\Delta Q_{v-1}| |\Delta \epsilon_v| \\
&= \sum_{v=1}^{\infty} \frac{P_v |t_v|}{p_v} q_v |\Delta \epsilon_v| \sum_{n=v+2}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \\
&= \sum_{v=1}^{\infty} \frac{P_v |t_v|}{p_v} \frac{q_v}{Q_{v+1}} |\Delta \epsilon_v| \\
&= O\left(\sum_{v=1}^{\infty} \frac{P_v}{p_v} \frac{q_v}{Q_v} |\Delta \epsilon_v| \mu_v\right)
\end{aligned}$$

$$= O\left(\sum_{v=1}^{\infty} |\Delta \epsilon_v| \mu_v\right), \text{ since } q_n P_n = O(p_n Q_n)$$

= O(1), by hypothesis of the theorem.

$$\begin{aligned}
Y_2 &= \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v|}{p_v} Q_v |\Delta^2 \epsilon_v| \\
&= \sum_{v=1}^{\infty} \frac{P_v}{P_v} Q_v |\Delta^2 \epsilon_v| |t_v| \sum_{n=v+2}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \\
&= O\left(\sum_{v=1}^{\infty} \frac{P_v}{p_v} Q_v |\Delta^2 \epsilon_v| \frac{\mu_v}{Q_{v+1}}\right) \\
&= O\left(\sum_{v=1}^{\infty} \frac{P_v}{p_v} |\Delta^2 \epsilon_v| \mu_v\right)
\end{aligned}$$

= O(1), by condition (2.3) of the theorem

$$Y_3 = \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v|}{p_v} |\Delta \varepsilon_{v+1}| q_v$$

$$= \sum_{v=1}^{\infty} \frac{P_v |t_v|}{p_v} |\Delta \varepsilon_{v+1}| q_v \sum_{n=v+2}^{\infty} \frac{q_n}{Q_n Q_{n-1}}$$

$$= O\left(\sum_{v=1}^{\infty} \frac{P_v}{p_v} \frac{|\Delta \varepsilon_{v+1}|}{Q_{v+1}} q_v \mu_v\right)$$

$$= O\left(\sum_{v=1}^{\infty} \frac{P_v}{p_v} \frac{q_v}{Q_v} |\Delta \varepsilon_{v+1}| \mu_v\right)$$

$$= O\left(\sum_{v=1}^{\infty} |\Delta \varepsilon_{v+1}| \mu_v\right), \text{ since } q_n P_n = O(p_n Q_n)$$

= O(1), by hypothesis of the theorem

$$Y_4 = \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-2} \frac{P_v |t_v|}{p_v} |\varepsilon_{v+2}| |\Delta q_v|$$

$$= \sum_{v=1}^{\infty} \frac{P_v |t_v|}{p_v} |\varepsilon_{v+2}| |\Delta q_v| \sum_{n=v+2}^{\infty} \frac{q_n}{Q_n Q_{n-1}}$$

$$= O\left(\sum_{v=1}^{\infty} \frac{P_v}{p_v} |\varepsilon_{v+2}| |\Delta q_v| \frac{1}{Q_{v+1}} \mu_v\right)$$

$$= O\left(\sum_{v=1}^{\infty} \frac{P_v}{p_v} \frac{q_v}{Q_v} |\varepsilon_{v+2}| \mu_v\right), \text{ since } \Delta q_v \leq q_v.$$

$$= O\left(\sum_{v=1}^{\infty} |\varepsilon_{v+2}| \mu_v\right), \text{ since } P_n q_n = O(p_n Q_n)$$

= O(1), by condition (2.2) of the theorem

So

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} J_{111} = O(1). \text{ This implies that } \Sigma_1 = O(1).$$

$$\text{Hence } \sum_{n=1}^{\infty} |T_n - T_{n-1}| < \infty.$$

This completes the proof of theorem 1.

Corollary. Let $\{p_n\}$ and $\{q_n\}$ be two positive monotonic non-increasing sequences such that $P_n q_n = O(p_n Q_n)$. If $\sum a_n = O(1)(\bar{N}, p_n)$, and if ε_n satisfies the conditions

$$(2.4) \quad \sum_{n=1}^{\infty} |\varepsilon_n| < \infty$$

$$(2.5) \quad \sum_{n=1}^{\infty} \frac{P_n}{p_{n+1}} |\Delta^2 \varepsilon_n| < \infty,$$

then the series $\sum a_n \varepsilon_n$ is summable $[\bar{N}, q_n]$

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